

A property shared by continuous linear functions and holomorphic functions

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Abstract: In this note, we continue to highlight some applications of Theorem 1 of [3]. Here is a sample: Let X be an open set in \mathbf{C}^n , Ω an open convex set in \mathbf{C} and $f, g : X \rightarrow \mathbf{C}$ two holomorphic functions such that $f(X) \cap \Omega \neq \emptyset$, $f(X) \setminus \Omega \neq \emptyset$ and $g(X) \subseteq \Omega$. Then, there exists a set A in $[0, 1]$ with the following properties:

- (a) for each $x \in X$, there exists $\lambda \in A$ such that $\lambda g(x) + (1 - \lambda)f(x) \in \Omega$;
- (b) for each finite set B in A , there exists $u \in X$ such that $\mu g(u) + (1 - \mu)f(u) \in \mathbf{C} \setminus \Omega$ for all $\mu \in B$.

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This paper is essentially a companion to [3].

In the sequel, the term "interval" means a non-empty connected subset of \mathbf{R} with more than one point.

For a multifunction $F : I \rightarrow 2^X$, as usual, for $A \subseteq I$ and $B \subseteq X$, we set

$$F(A) = \bigcup_{x \in A} F(x) ,$$

$$F^-(B) = \{\lambda \in I : F(\lambda) \cap B \neq \emptyset\}$$

and

$$F^+(B) = \{\lambda \in I : F(\lambda) \subseteq B\} .$$

When I is an interval, F is said to be non-decreasing (resp. non-increasing) with respect to the inclusion if $F(\lambda) \subseteq F(\mu)$ (resp. $F(\mu) \subseteq F(\lambda)$) for all $\lambda, \mu \in I$, with $\lambda < \mu$.

Furthermore, let Y be a non-empty set and \mathcal{F} a family of subsets of Y . We say that \mathcal{F} has the compactness-like property if every subfamily of \mathcal{F} satisfying the finite intersection property has a non-empty intersection. An obvious (but useful) remark is that if \mathcal{F} has the compactness-like property and $\Gamma_0 \neq \emptyset$ is a member of \mathcal{F} , then the family $\{\Gamma \cap \Gamma_0\}_{\Gamma \in \mathcal{F}}$ has the compactness-like property too.

In [3], we established the following general result:

THEOREM A. - *Let X be a non-empty set, $I \subseteq \mathbf{R}$ an interval and $F : I \rightarrow 2^X$ a multifunction such that, for each $x \in X$, the set $X \setminus F^-(x)$ is an interval open in I . Moreover, assume that, for some $\lambda_0 \in I$, with $F(\lambda_0) \neq \emptyset$, and for some set $D \subseteq I$ dense in I , the family $\{F(\lambda) \cap F(\lambda_0)\}_{\lambda \in D}$ has the compactness-like property.*

Under such hypotheses, there exists a compact interval $[a^, b^*] \subseteq I$ such that either $a^* \geq \lambda_0$, $(F(a^*) \cap F(\lambda_0)) \setminus F([a^*, b^*]) \neq \emptyset$ and $F_{[a^*, b^*]}$ is non-decreasing with respect to the inclusion, or $b^* \leq \lambda_0$, $(F(b^*) \cap F(\lambda_0)) \setminus F([a^*, b^*]) \neq \emptyset$ and $F_{[a^*, b^*]}$ is non-increasing with respect to the inclusion. In particular, the first (resp. second) occurrence is true when $\lambda_0 = \inf I$ (resp. $\lambda_0 = \sup I$). Furthermore, if, for some neighbourhood U of λ_0 in I , one has $\bigcap_{\lambda \in U} F(\lambda) \neq \emptyset$, then in the first (resp. second) occurrence one has $a^* > \lambda_0$ (resp. $b^* < \lambda_0$).*

REMARK 1. - We want to remark that the very final part of the conclusion is not present in the formulation given in [1]. Such a further information follows immediately from the proof. Hence, we refer the reader to [3] and limit ourselves here to highlight the relevant point only, keeping the notations of the proof in [1]. So, assume that, for some $\delta > 0$, one has $\cap_{\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta] \cap I} F(\lambda) \neq \emptyset$. Pick $\tilde{x} \in \cap_{\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta] \cap I} F(\lambda)$. Hence, either $\alpha(\tilde{x}) \geq \lambda_0 + \delta$ or $\beta(\tilde{x}) \leq \lambda_0 - \delta$. If $\alpha(\tilde{x}) \geq \lambda_0 + \delta$ (resp. $\beta(\tilde{x}) \leq \lambda_0 - \delta$), we know that the first (resp. second) occurrence of the conclusion is true with $a^* = \sup_X \alpha$ (resp. $b^* = \inf_X \beta$), and the claim follows.

Within the project of a systematic use of Theorem A, we obtained, in particular, Theorem 3 of [3].

Our aim in the present paper is to give an extension of this latter result to multifunctions (Theorem 2) and then an application which highlights the property mentioned in the title (Theorem 3).

In the sequel, Y is a real or complex Hausdorff locally convex topological vector space, E is a closed subset of Y such that $Y \setminus E$ is convex, $I \subseteq \mathbf{R}$ is an interval, $\lambda_0 \in I$ and $G : X \times I \rightarrow 2^Y$ is a given multifunction. The symbol ∂ stands for boundary.

We recall a further notion on multifunctions and refer to [1], [2] for other classical notions.

Namely, a multifunction $\Phi : I \rightarrow 2^Y$ is said to be concave if one has

$$\Phi(t\lambda + (1-t)\mu) \subseteq t\Phi(\lambda) + (1-t)\Phi(\mu)$$

for all $t \in [0, 1]$, $\lambda, \mu \in I$.

Our main result is as follows:

THEOREM 1. - *Let the following assumptions hold:*

- (i₁) *for each $x \in X$, the multifunction $G(x, \cdot)$ is concave and continuous in I , and $G(x, \lambda) \subseteq Y \setminus E$ for some $\lambda \in I$;*
- (i₂) *the set $\{x \in X : G(x, \lambda_0) \cap \text{int}(E) \neq \emptyset\}$ is non-empty ;*
- (i₃) *there exists a set $D \subseteq I$ dense in I such that, for every set $A \subseteq D$ for which*

$$\bigcap_{\lambda \in B \cup \{\lambda_0\}} \{x \in X : G(x, \lambda) \cap E \neq \emptyset\} \neq \emptyset$$

for each finite set $B \subseteq A$, one has

$$\bigcap_{\lambda \in A \cup \{\lambda_0\}} \{x \in X : G(x, \lambda) \cap E \neq \emptyset\} \neq \emptyset .$$

Then, there exist a compact interval $[a^, b^*] \subseteq I$ and a point $x^* \in X$, with $G(x^*, \lambda_0) \cap E \neq \emptyset$ and $G(x^*,]a^*, b^*]) \subseteq Y \setminus E$, such that, if we put*

$$V = \bigcup_{\lambda \in]a^*, b^*[} \{x \in X : G(x, \lambda) \subseteq Y \setminus E\} ,$$

at least one of the following holds:

- (p₁) *$a^* > \lambda_0$, $G(x^*, a^*) \cap E \neq \emptyset$ and*

$$G(V, a^*) \cap E \subseteq \partial G(V, a^*) \cap \partial E ;$$

- (p₂) *$b^* < \lambda_0$, $G(x^*, b^*) \cap E \neq \emptyset$ and*

$$G(V, b^*) \cap E \subseteq \partial G(V, b^*) \cap \partial E .$$

In particular, (p₁) (resp. (p₂)) holds when $\lambda_0 = \inf I$ (resp. $\lambda_0 = \sup I$).

PROOF. Consider the multifunction $F : I \rightarrow 2^X$ defined by

$$F(\lambda) = \{x \in X : G(x, \lambda) \cap E \neq \emptyset\}$$

for all $\lambda \in I$. Note that, by (i_2) , there is $\tilde{x} \in X$ such that $G(\tilde{x}, \lambda_0) \cap \text{int}(E) \neq \emptyset$. Then, since $G(\tilde{x}, \cdot)$ is lower semicontinuous, there is a neighbourhood U of λ_0 in I such that $G(\tilde{x}, \lambda) \cap \text{int}(E) \neq \emptyset$ for all $\lambda \in U$. Therefore, $\tilde{x} \in \cap_{\lambda \in U} F(\lambda)$. In view of (i_3) , the family $\{F(\lambda) \cap F(\lambda_0)\}_{\lambda \in D}$ has the compactness-like property. Moreover, by (i_1) , for each $x \in X$, the set $I \setminus F^-(x)$ (that is $\{\lambda \in I : G(x, \lambda) \subseteq Y \setminus E\}$) is non-empty, convex (since $G(x, \cdot)$ is concave and $Y \setminus E$ is convex) and open in I (since $G(x, \cdot)$ is upper semicontinuous and $Y \setminus E$ is open). Hence, the multifunction F satisfies the hypotheses of Theorem A. Consequently, there exists a compact interval $[a^*, b^*] \subseteq I$ such that either $a^* > \lambda_0$, $(F(a^*) \cap F(\lambda_0)) \setminus F(]a^*, b^*]) \neq \emptyset$ and $F_{]a^*, b^*]}$ is non-decreasing with respect to the inclusion, or $b^* < \lambda_0$, $(F(b^*) \cap F(\lambda_0)) \setminus F(]a^*, b^*]) \neq \emptyset$ and $F_{]a^*, b^*]}$ is non-increasing with respect to the inclusion. For instance, assume that the first alternative holds. Pick $x^* \in (F(a^*) \cap F(\lambda_0)) \setminus F(]a^*, b^*])$. So, $G(x^*, \lambda_0) \cap E \neq \emptyset$, $G(x^*, a^*) \cap E \neq \emptyset$ and $G(x^*,]a^*, b^*]) \subseteq Y \setminus E$. First, let us prove that $G(V, a^*) \cap E \subseteq \partial E$. So, let $y \in G(V, a^*) \cap E$. Arguing by contradiction, assume that $y \in \text{int}(E)$. Let $x \in V$ be such that $y \in G(x, a^*)$. Also, let $\mu \in]a^*, b^*]$ be such that $G(x, \mu) \subseteq Y \setminus E$. Since $G(x, a^*) \cap \text{int}(E) \neq \emptyset$ and $G(x, \cdot)$ is lower semicontinuous, we can find $\lambda \in]a^*, \mu[$ so that $G(x, \lambda) \cap \text{int}(E) \neq \emptyset$. But then $X \setminus F(\mu) \subseteq X \setminus F(\lambda)$ and so $G(x, \lambda) \subseteq Y \setminus E$, a contradiction. Next, let us prove that $G(V, a^*) \cap E \subseteq \partial G(V, a^*)$. Fix $z \in G(V, a^*) \cap E$. Arguing by contradiction again, assume that $z \in \text{int}(G(V, a^*))$. We already know that $z \in \partial E$ and so $z \in \partial(Y \setminus E)$. Since $Y \setminus E$ is open and convex, a classical separation theorem ensures the existence of a non-zero continuous linear functional $\varphi : Y \rightarrow \mathbf{R}$ such that $\varphi(z) < \varphi(u)$ for all $u \in Y \setminus E$. Therefore, the set $\varphi^{-1}(]-\infty, \varphi(z)])$ is contained in E . Moreover, it is open and meets $\text{int}(G(V, a^*))$ since, otherwise, z would be a local minimum of φ , which is impossible since φ is linear. Consequently, the set $\varphi^{-1}(]-\infty, \varphi(z)]) \cap \text{int}(G(V, a^*))$ would be non-empty, open and contained in $G(V, a^*) \cap E$. So, by what we have already seen, we would have $\varphi^{-1}(]-\infty, \varphi(z)]) \cap \text{int}(G(V, a^*)) \subseteq \partial E$, against the fact that $\text{int}(\partial E) = \emptyset$ since E is closed. So (p_1) is proved. If the second alternative above holds, we obtain (p_2) by means of analogous reasonings. \triangle

REMARK 2. - Notice that the mere lower semicontinuity of $G(x, \cdot)$ is not enough for the validity of Theorem 1. To see this, take $Y = \mathbf{R}$, $E = \mathbf{R} \setminus]-1, 1[$, $I = [0, 1]$ and $G(x, \lambda) =]-1, 1[-\lambda$ for all $(x, \lambda) \in X \times [0, 1]$. Clearly, each assumptions of Theorem 1, but the upper semicontinuity of $G(x, \cdot)$, is satisfied. However, neither (p_1) nor (p_2) holds since the values of G are open.

REMARK 3. - Clearly, if we give up the better information $a^* > \lambda_0$ (resp. $b^* < \lambda_0$), the continuity of $G(x, \cdot)$ can be weakened to upper semicontinuity and, at the same time, (i_2) can be weakened to $\{x \in X : G(x, \lambda_0) \cap E \neq \emptyset\} \neq \emptyset$.

The above-mentioned extension of Theorem 3 of [3] obtained via Theorem 1 is as follows:

THEOREM 2. - Let $0 \in I$ and let Φ, Ψ be two multifunctions from X into Y , with non-empty compact values. Assume that:

- (h_1) for each $x \in X$, there exists $\lambda \in I$ such that $\Phi(x) + \lambda\Psi(x) \subseteq Y \setminus E$;
- (h_2) the set $\Phi^-(\text{int}(E))$ is non-empty ;
- (h_3) there exists a set $D \subseteq I$ dense in I such that, for every set $A \subseteq D$ for which

$$\bigcap_{\lambda \in B \cup \{0\}} (\Phi + \lambda\Psi)^-(E) \neq \emptyset$$

for each finite set $B \subseteq A$, one has

$$\bigcap_{\lambda \in A \cup \{0\}} (\Phi + \lambda\Psi)^-(E) \neq \emptyset .$$

Then, there exist a compact interval $[a^*, b^*] \subseteq I$ and a point $x^* \in \Phi^-(E)$, with

$$\bigcup_{\lambda \in]a^*, b^*]} (\Phi(x^*) + \lambda\Psi(x^*)) \subseteq Y \setminus E ,$$

such that, if we put

$$V = \bigcup_{\lambda \in]a^*, b^*]} (\Phi + \lambda\Psi)^+(Y \setminus E) ,$$

at least one of the following holds:

(q_1) $a^* > 0$, $(\Phi(x^*) + a^*\Psi(x^*)) \cap E \neq \emptyset$ and

$$(\Phi + a^*\Psi)(V) \cap E \subseteq \partial(\Phi + a^*\Psi)(V) \cap \partial E ;$$

(q_2) $b^* < 0$, $(\Phi + b^*\Psi)(V) \cap E \neq \emptyset$ and

$$(\Phi + b^*\Psi)(V) \cap E \subseteq \partial(\Phi + b^*\Psi)(V) \cap \partial E .$$

In particular, (q_1) (resp. (q_2)) holds when $0 = \inf I$ (resp. $0 = \sup I$) .

PROOF. Apply Theorem 1, with $\lambda_0 = 0$, to the multifunction $G : X \times I \rightarrow 2^Y$ defined by

$$G(x, \lambda) = \Phi(x) + \lambda\Psi(x)$$

for all $(x, \lambda) \in X \times I$. In particular, note that the multifunction $G(x, \cdot)$ is continuous since both $\Phi(x), \Psi(x)$ are compact. \triangle

REMARK 4. - According to Remark 3, if, instead of (h_2), we simply assume that $\Phi^-(E) \neq \emptyset$, then (q_1) (resp. (q_2)) holds with $a^* \geq 0$ (resp. $b^* \leq 0$).

Here is the property mentioned in the title:

THEOREM 3. - Let $f, g : X \rightarrow Y$ be two functions such that $f(X) \cap E \neq \emptyset$, $f(X) \setminus E \neq \emptyset$ and $g(X) \subseteq Y \setminus E$. Moreover, suppose that one of the two following sets of assumptions holds:

(k_1) X is an open set in a Banach space S , $\dim(Y) < \infty$ and f, g are the restrictions to X of two continuous linear functions from S into Y ;

(k_2) X is an open set in \mathbf{C}^n , $Y = \mathbf{C}$ and f, g are holomorphic in X .

Then, for every set $D \subseteq [0, 1]$ dense in $[0, 1]$, with $0 \in D$, there exists a set $A \subseteq D$ with the following properties:

(r_1) for every $x \in X$, there exists $\lambda \in A$ such that

$$\lambda g(x) + (1 - \lambda)f(x) \in Y \setminus E ;$$

(r_2) for every finite set $B \subseteq A$, there exists $u \in X$ such that

$$\mu g(u) + (1 - \mu)f(u) \in E$$

for all $\mu \in B$.

PROOF. Arguing by contradiction, assume that the conclusion is false. So, assume that there is a set $D \subseteq [0, 1]$ dense in $[0, 1]$, with $0 \in D$, such that, for every set $A \subseteq D$ for which

$$\bigcap_{\lambda \in A} (\lambda g + (1 - \lambda)f)^{-1}(E) \neq \emptyset$$

for each finite set $B \subseteq A$, one has

$$\bigcap_{\lambda \in A} (\lambda g + (1 - \lambda)f)^{-1}(E) \neq \emptyset .$$

This means that the family $\{(\lambda g + (1 - \lambda)f)^{-1}(E)\}_{\lambda \in D}$ has the compactness-like property. Then, since $0 \in D$, the family $\{(\lambda g + (1 - \lambda)f)^{-1}(E) \cap f^{-1}(E)\}_{\lambda \in D}$ has the compactness-like property too. Hence, if we take $I = [0, 1]$, $\Phi = f$ and $\Psi = g - f$, assumption (h_3) of Theorem 2 is satisfied. Moreover, by assumption, $f^{-1}(E) \neq \emptyset$ and, for each $x \in X$, there is $\lambda \in [0, 1]$ such that $f(x) + \lambda(g(x) - f(x)) \in Y \setminus E$ (actually, we can take $\lambda = 0$ if $f(x) \in Y \setminus E$, and $\lambda = 1$ if $f(x) \in E$). Hence, by Theorem 2 (recalling Remark 4), there exist an interval $[a^*, b^*] \subseteq [0, 1]$ and a point $x^* \in f^{-1}(E)$, with

$$w := a^* g(x^*) + (1 - a^*)f(x^*) \in E$$

and

$$\lambda g(x^*) + (1 - \lambda)f(x^*) \in Y \setminus E$$

for all $\lambda \in]a^*, b^*[$, such that, if we put

$$V = \bigcup_{\lambda \in]a^*, b^*[} (\lambda g + (1 - \lambda)f)^{-1}(Y \setminus E) ,$$

we have

$$(a^*g + (1 - a^*)f)(V) \cap E \subseteq \partial(a^*g + (1 - a^*)f)(V) \cap \partial E . \quad (1)$$

Now, assume that (k_1) holds. Continue to denote by the same symbols the continuous linear extensions of f, g to S . Set

$$T = (a^*g + (1 - a^*)f)(S) .$$

Since $a^*g + (1 - a^*)f$ is linear and continuous and $\dim(T) < \infty$, by the open mapping theorem, the set $(a^*g + (1 - a^*)f)(V)$ is open in T since V is open in S . Then, since $x^* \in V$, there exists an open neighbourhood W of w in Y such that

$$W \cap T \subseteq (a^*g + (1 - a^*)f)(V) .$$

From this, it follows that

$$W \cap T \cap E \subseteq (a^*g + (1 - a^*)f)(V) \cap E$$

and so, in view of (1),

$$W \cap T \cap E \subseteq \partial E . \quad (2)$$

Then, since $w \in W \cap T \cap E$, we have $w \in \partial(Y \setminus E)$. Let $\varphi : Y \rightarrow \mathbf{R}$ be a non-zero continuous linear functional such that $\varphi(u) < \varphi(w)$ for all $u \in Y \setminus E$. By assumption, there is $\tilde{x} \in X$ such that $f(\tilde{x}) \in Y \setminus E$. Set

$$\tilde{w} = a^*g(\tilde{x}) + (1 - a^*)f(\tilde{x}) .$$

Hence, $\varphi(\tilde{w}) < \varphi(w)$ as $\tilde{w} \in Y \setminus E$. Now, fix $\lambda < 0$ so that $w + \lambda(\tilde{w} - w) \in W$. Then, observing that $Y \setminus \overline{Y \setminus E} = \text{int}(E)$ and that $\varphi(w + \lambda(\tilde{w} - w)) > \varphi(w)$, we have

$$w + \lambda(\tilde{w} - w) \in \text{int}(E)$$

which contradicts (2).

Now, suppose that (k_2) holds. Since $a^*g + (1 - a^*)f$ is holomorphic and not constant (note that $w \in \partial E$ and $\tilde{w} \in \mathbf{C} \setminus E$), and V is open in \mathbf{C}^n , by a classical result, the set $(a^*g + (1 - a^*)f)(V)$ is open in \mathbf{C} and this is against (1). The proof is complete. \triangle

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